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Synthesis of Waffle Plates with Multiple Rib Sizes

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The relative efficiency of minimum-weight designs of waffle plates having multiple rib sizes in each stiffening direction is investigated, and compared to that of basic waffle and honeycomb-core sandwich construction. The behavior of orthogonally stiffened rectangular plates having two sets of rib sizes, and subjected to uniaxial compression, is studied in detail. Synthesis is accomplished by formulating the design requirements in terms of a mathematical programing problem, and by using a penalty function optimization algorithm suggested by Fiacco and McCormick. Results of examples indicate that a significant improvement in waffle efficiency may be achieved through incorporation of successively larger sizes of ribs at appropriate intervals, to the point where waffle construction may become competitive with sandwich construction at low loading intensities.

Nomenclature

a = plate length in the x direction
b = plate length in the y direction
c_i = location of secondary ribs in the x direction
C_1,C_2,C_3,C_4 = analysis constants
C = $Et/(1 - \nu^2)$ = in-plane stiffness of skin
d,x_2 = primary rib spacing
d_x = spacing of secondary ribs oriented in the y direction
d_y = spacing of secondary ribs oriented in the x direction
D = $Et^3/12(1 - \nu^2)$ = skin bending stiffness
$\mathbf{D}$ = matrix of plate bending stiffness
e_j = location of secondary ribs in the y direction
e_y = eccentricity of loading from middle surface
E = modulus of elasticity
$f(\mathbf{X})$ = objective function
$g_i(\mathbf{X})$ = constraints
h_1,x_4 = height of primary ribs
h_2,x_6 = height of secondary ribs
I = moment of inertia of primary ribs about middle surface of skin
I_i = moment of inertia of secondary ribs about middle surface
k = plate buckling coefficient
$\mathbf{K}$ = stiffness matrix
m = integer in deflection series representation for w
n = integer in deflection series
$N_i(\mathbf{X})$ = plate failure loads
N_y = applied load in the y direction
N'_i = component failure loads
N^*_i = component failure loads for plate with only primary ribs
$Q(\mathbf{X},\mathbf{X}^k)$ = penalty function
$\mathbf{S}$ = stability matrix
t,x_1 = skin thickness
t_1,x_3 = primary rib thickness
t_2,x_5 = secondary rib thickness
$\bar{t}$ = equivalent thickness for loading
u,v,w = plate displacements in the x,y, and z directions, respectively
$\mathbf{W}$ = vector of generalized bending displacements
w_{mn} = displacement amplitudes
$\mathbf{W}_{mn}$ = vector of displacement amplitudes
$\mathbf{X}$ = vector of design variables
$\mathbf{X}^k$ = solution to unconstrained problem k
$\epsilon_x,\epsilon_y,\epsilon_{xy}$ = strains
Π = energy
ν = Poisson's ratio
$\sigma_x,\sigma_y,\sigma_{xy}$ = stresses
$\kappa_x,\kappa_y,\kappa_{xy}$ = curvatures
ρ = material density

Introduction

SANDWICH construction is recognized as the most efficient means, from a weight standpoint, of transmitting compressive loads of relatively low intensity. However, elaborate and costly inspection methods are often required to insure the integrity of the sandwich. In addition, much of the inherent efficiency may be sacrificed due to the complexity associated with edge attachments and joints. Integrally stiffened "waffle" structures, although inherently less efficient,

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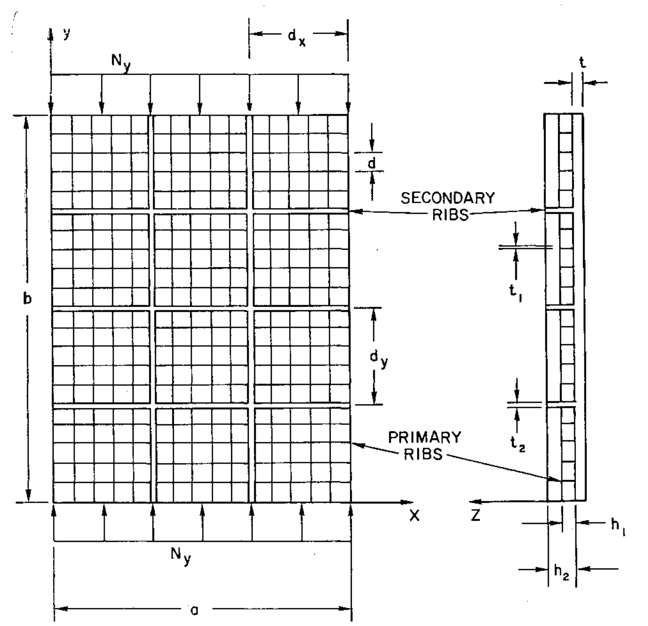

Fig. 1 Configuration of waffle plate with primary and secondary ribs.

offer an attractive alternative to sandwich structures, particularly in view of modern advances in machining and forming technology.

Regardless of stiffener geometry or orientation, all known applications of waffle construction apparently employ only one basic size of stiffener in any direction. Under given loads, the basic stiffeners must then be proportioned and spaced so as to prevent premature local buckling of sheet and stiffeners, as well as the occurrence of a fundamental gross buckling mode for the entire plate.

If, now, larger "secondary" ribs are added to the basic waffle plate, as shown in Fig. 1, a twofold effect will be observed. First, the gross buckling load will be increased, possibly accompanied by a change in fundamental mode shape. Second, a possible new failure mode is introduced. This new failure mode involves buckling of that portion of the stiffened plate enclosed by secondary ribs. This relatively minor alteration in configuration is thus seen to have a significant effect on plate behavior. Conversely, it is logical to anticipate that, for given loads, a more efficient design may be obtained if all plate parameters are re-evaluated with the new behavior, which results from the addition of secondary ribs, taken into account. Furthermore, if the over-all configuration and loading should warrant it, subsequent sets of successively larger ribs may be incorporated at appropriate intervals.

The primary purpose of this paper is to initiate an investigation of the effect of the aforementioned "pyramiding" of rib sizes on the optimum (minimum-weight) design of integrally stiffened waffle plates. In order to demonstrate the potentialities of this structural concept, the present paper will be limited to a study of uniaxially loaded rectangular plates having orthogonal stiffeners parallel to the plate edges.

The optimum plate configuration will be obtained by formulating the design requirements in terms of a nonlinear programing problem. The design of orthogonally stiffened rectangular plates having basic (primary) stiffeners only, and subjected to combined loading and multiple distinct loading conditions, has been previously treated as an inequality constrained optimization problem by Schmit, Kicher, and Morrow.[1]‡ Similar techniques have also been used by Kicher[2] for the design of integrally stiffened cylinders. More recently, Schmit et al.[3] have used a penalty function approach to transform the stiffened cylinder design from a constrained optimization problem to an unconstrained optimization problem. This latter approach has certain computational advantages over the former. In the present paper, a variation on the interior penalty function method of Fiacco and McCormick suggested in Ref. 4 is employed.

‡ Ref. 19 utilizes a random sampling procedure to re-evaluate some of the designs obtained in Ref. 1. The authors wish to thank one of the reviewers for bringing this reference to their attention.

Problem Statement

In order to gain some insight into the effect of "pyramided" stiffeners on waffle plate efficiency, the design problem will be formulated for the type of structure illustrated in Fig. 1. Plate edges will be taken as simply supported, and behavior will be assumed to be elastic, with limitations imposed by the possibility of buckling below or at the yield stress σ_y. Modulus of elasticity E, Poisson's ratio ν, and material density ρ will be assumed to be specified constants. Plate dimensions, a and b, as well as load N_y, are also initially specified quantities.

The spacing and size of primary ribs will be assumed equal in both the x and y directions in order to keep the number of design variables to a minimum. For this reason, the secondary rib spacings, d_x and d_y, will also be treated as prespecified quantities, and their effect on the final design will be determined parametrically.

The synthesis problem then reduces to one containing six variables; skin thickness t, spacing of primary ribs d, thickness of primary ribs t_1, height of primary ribs h_1, thickness of secondary ribs t_2, and height of secondary ribs h_2. Each possible design will be completely characterized by the determination of the components of a solution vector $\mathbf{X}$, where

$$\mathbf{X}^T = \{t\, d\, t_1\, h_1\, t_2\, h_2\} \equiv \{x_i\},\ i = 1,2,\ldots,6 \tag{1}$$

and where $x_1 = t$, $x_2 = d$, etc. (Both notations will be used interchangeably in subsequent sections for convenience and clarity.)

There are six distinct limitations on plate behavior which result in constraints on the choice of design variables. For any design, the applied load must be no greater than 1) the load, $N_1(\mathbf{X})$, at which gross buckling of the entire plate occurs, 2) the load, $N_2(\mathbf{X})$, at which buckling of the stiffened panels enclosed by secondary ribs occurs, 3) the load, $N_3(\mathbf{X})$, at which buckling of the unstiffened skin between primary ribs occurs, 4) the load, $N_4(\mathbf{X})$, at which buckling of primary ribs occurs, 5) the load, $N_5(\mathbf{X})$, at which buckling of secondary ribs occurs, 6) the load, $N_6(\mathbf{X})$, at which the yield stress is reached in either skin or stiffeners.

Acceptable designs will thus be obtained whenever

$$g_i(\mathbf{X}) \equiv \frac{N_i(\mathbf{X}) - N_y}{N_y} \geqq 0,\ i = 1,2,\ldots,6 \tag{2}$$

In addition to these "behavioral" constraints, there are several "side" constraints required to insure obtaining realistic designs. First, all design variables must be nonnegative. This requirement leads to six more constraints of the form

$$g_{i+6}(\mathbf{X}) \equiv x_i \geqq 0,\ i = 1,2,\ldots,6 \tag{3}$$

Second, the spacing of primary ribs should not exceed the spacing of secondary ribs. Therefore

$$g_{13}(\mathbf{X}) \equiv d_x - x_2 \geqq 0 \tag{4a}$$

$$g_{14}(\mathbf{X}) \equiv d_y - x_2 \geqq 0 \tag{4b}$$

Note, again, that d_x and d_y are prespecified quantities defined in Fig. 1.

Finally, stiffener spacing must be greater than the thicknesses of corresponding stiffeners, or

$$g_{15}(\mathbf{X}) \equiv x_2 - x_3 \geqq 0 \tag{5a}$$

$$g_{16}(\mathbf{X}) \equiv d_x - x_5/2 \geqq 0 \tag{5b}$$

$$g_{17}(\mathbf{X}) \equiv d_y - x_5/2 \geqq 0 \tag{5c}$$

The objective of the synthesis process is the minimum-weight design of the waffle plate. The "objective function" is therefore

$$w(\mathbf{X}) = \rho ab[f(\mathbf{X})] \tag{6a}$$

where

$$f(\mathbf{X}) \doteq t + \frac{t_1h_1}{d}\left(2 - \frac{t_1}{d}\right) + \frac{t_2h_2}{ab} \times \left[\frac{b(a - d_x)}{d_x} + \frac{a(b - d_y)}{d_y} - \frac{t_2(a - d_x)(b - d_y)}{d_xd_y}\right] \tag{6b}$$

The design problem can now be restated in the form

$$\text{Minimize:}\ \ f(\mathbf{X}) \tag{7a}$$

$$\text{Subject to:}\ \ g_i(\mathbf{X}) \geqq 0,\ i = 1,2,\ldots,17 \tag{7b}$$

Clearly, additional quantities may be treated as design variables, and multiple biaxial load conditions may be considered, but it is not within the scope of this paper to treat the most general synthesis problem of this type.

Analysis of Waffle Plate Behavior

For any set of design variables, a suitable method of analysis must be used in conjunction with the optimization algorithm to determine whether or not any of the aforementioned behavioral constraints are violated. Means for predicting various failure loads, $N_i(\mathbf{X})$, will now be developed. In general, all buckling modes will be assumed to be independent.

1. Gross Buckling

It will be assumed that the predominant strain energy in the plate during buckling is due to bending of the plate-stiffener combination, and also that the spacing of primary ribs is small compared to over-all plate dimensions. This latter assumption allows the effect of the primary ribs to be "smeared out."[5]

Denoting displacement in the z direction by w, and following Timoshenko,[6] the strain-energy in bending of plate and primary stiffeners is given by

$$\Pi_P = \frac{1}{2}\int_0^a\int_0^b \mathbf{W}^T\mathbf{D}\mathbf{W}dydx \tag{8a}$$

where

$$\mathbf{W}^T = \{w_{,xx}\ w_{,yy}\ 2w_{,xy}\} \tag{8b}$$

and

$$\mathbf{D} = \begin{bmatrix} D + I & D\nu & 0 \\ D\nu & D + I & 0 \\ 0 & 0 & \frac{1}{2}D(1 - \nu) \end{bmatrix} \tag{8c}$$

where D is the skin stiffness given by

$$D = Et^3/12(1 - \nu^2) \tag{8d}$$

and I is the moment of inertia of the primary ribs about the middle surface of the skin, and

$$I = [Et_1/d](\tfrac{1}{3}h_1^3 + \tfrac{1}{2}h_1^2t + \tfrac{1}{4}h_1t^2) \tag{8e}$$

The strain energy of the secondary ribs in bending is

$$\Pi_R = \frac{1}{2}E\sum_i I_i\int_0^a (w_{,xx})^2{}_{y=c_i}\,dx + \frac{1}{2}E\sum_j I_j\int_0^b (w_{,yy})^2{}_{x=e_j}\,dy \tag{9a}$$

where the i ribs lie in the x direction at a distance of c_i from the x axis, and the j ribs lie in the y direction at a distance of e_j from the y axis. The i summation in the preceding expression is over all secondary ribs in the x direction, and the j summation is over all secondary ribs in the y direction. Assuming the centroid of the stiffened plate system to be approximately at the surface of the skin,

$$I_i = I_j = \tfrac{1}{3}t_2h^3{}_2 \tag{9b}$$

The work done by the external load during bending of the plate and primary stiffeners is

$$\Pi_{WP} = \frac{1}{2}\int_0^a\int_0^b N_1(w_{,y})^2dydx \tag{10}$$

For the examples to be considered in this study, the secondary ribs are spaced at intervals which do not permit their effect to be "smeared out" as in the case of the primary ribs. As shown in Ref. 6, the work done by the external load during bending of these ribs must be considered separately. This work is given by

$$\Pi_{WR} = \frac{1}{2}\sum_j\left(\frac{h_2t_2}{\bar{t}}\right)_j\int_0^b N_1(w_{,y})^2{}_{x=e_j}\,dy \tag{11a}$$

where

$$\bar{t} = t + t_1h_1/d + t_2h_2/d_x \tag{11b}$$

For the case of simple edge supports, the bending deflection can be represented by

$$w = \sum_n\sum_m w_{mn}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b} \tag{12}$$

Using this in the aforementioned relationships, Eqs. (8a, 9a, 10, and 11), gives

$$\Pi_P = \frac{\pi^4ab}{8}\sum_n\sum_m w_{mn}{}^2\left\{D\left[\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^4\right]^2 + \left[\left(\frac{n}{a}\right)^4 + \left(\frac{m}{b}\right)^2\right]I\right\} \tag{13a}$$

$$\Pi_R = \sum_i\left\{\frac{\pi^4EI_i}{4a^3}\sum_n n^4\left(\sum_m w_{mn}\sin\frac{m\pi c_i}{b}\right)^2\right\} + \sum_j\left\{\frac{\pi^4EI_j}{4b^3}\sum_m m^4\left(\sum_n w_{mn}\sin\frac{n\pi e_j}{a}\right)^2\right\} \tag{13b}$$

$$\Pi_{WP} = \frac{ab}{8}N_1\sum_n\sum_m w_{mn}{}^2\frac{m^2\pi^2}{b^2} \tag{13c}$$

$$\Pi_{WR} = \sum_j\left\{\frac{\pi^2}{4b}N_1\left(\frac{t_2h_2}{\bar{t}}\right)_j\sum_m m^2\left(\sum_n w_{mn}\sin\frac{n\pi e_j}{a}\right)\right\} \tag{13d}$$

The total potential energy of the system is the sum of all the preceding energy forms. The Theorem of Minimum Potential Energy states that a necessary condition for an equilibrium configuration is that the first variation of this sum must vanish. Taking the variation with respect to w_{mn} for all m and n leads to a set of equations of the form

$$\mathbf{KW_{mn}} = N_1\mathbf{SW_{mn}} \tag{14}$$

where $\mathbf{K}$ denotes stiffness and $\mathbf{S}$ is a stability matrix.[7,8]

Equation (14) constitutes an eigenvalue problem which can be solved for $n \times m$ critical loads N_1, and corresponding eigenvectors $\mathbf{W_{mn}}$. In the work reported here, maximum values of both n and m were taken as 5 and an iterative procedure[9] utilizing matrix decomposition[10] was used to find the lowest critical load.

In the process of designing the stiffened plate system by mathematical programing methods, many candidate designs must be analyzed and evaluated before an optimum design will be obtained. Since the previously described method for calculating N_1 corresponding to gross failure requires an iterative procedure involving the solution of a set of simultaneous equations, it is rather time consuming computationally. Therefore, in the design scheme, a modification of this method was actually used. It was assumed that only one mode shape is predominant in the gross buckling of the stiffened plate, therefore allowing n and m to take on only one value each. This reduces the system of equations obtained from (14) to only one equation which may then be solved in closed form for N_1. A range of n and m values were considered in this way, and those values of n and m which gave the lowest value of N_1 were considered to constitute the solution. Since the plate-stiffener system is essentially forced by this procedure to assume a displacement pattern other than the one which will truly minimize the potential energy, the values calculated by using this approximation are somewhat higher than those which will be found by using a superposition of mode shapes. In the actual design scheme, however, the final (optimum) design was checked for gross buckling strength using the more rigorous method. Results of the approximate method of analysis were found to be adequate for design purposes.

2. Panel Buckling

As shown in Fig. 1, the secondary stiffeners divide the plate into panels which are reinforced by only the primary ribs. These panels are assumed to behave as stiffened plates which will buckle under the action of the compressive load N_2. It may also be conservatively assumed that such panels act as if they are simply supported over the length d_y and width d_x. The analysis follows that given by McElman,[5] with the exception that only linear strain-displacement relations are used here. Also, the effect of twisting of the stiffeners is neglected.

The strain energy of a rectangular plate of uniform thickness t, length d_y, and width d_x is

$$\Pi_S = \frac{1}{2}\int_0^{d_x}\int_0^{d_y}\int_{-t/2}^{t/2} (\sigma_x{}^s\epsilon_x{}^s + \sigma_y{}^s\epsilon_y{}^s + \sigma_{xy}{}^s\epsilon_{xy}{}^s)dzdydx \tag{15}$$

where $\sigma_x{}^s$, $\sigma_y{}^s$, and $\sigma_{xy}{}^s$ are the skin stresses and $\epsilon_x{}^s$, $\epsilon_y{}^s$, and $\epsilon_{xy}{}^s$ the corresponding strains. If it is assumed that the plate, when loaded, is in a state of plane stress, the stress-strain relations are

$$\sigma_x{}^s = [E/(1 - \nu^2)](\epsilon_x{}^s + \nu\epsilon_y{}^s) \tag{16a}$$

$$\sigma_y{}^s = [E/(1 - \nu^2)](\epsilon_y{}^s + \nu\epsilon_x{}^s) \tag{16b}$$

$$\sigma_{xy}{}^s = [E/2(1 + \nu)]\epsilon_{xy}{}^s \tag{16c}$$

Using these relations in Eq. (15), leads to

$$\Pi_S = \frac{E}{2(1-\nu^2)}\int_0^{d_x}\int_0^{d_y}\int_{-t/2}^{t/2}\left[(\epsilon_x{}^s)^2 + (\epsilon_y{}^s)^2 + 2\nu\epsilon_x{}^s\epsilon_y{}^s + \frac{1}{2}(1-\nu)(\epsilon_{xy}{}^s)^2\right]dzdydx \tag{17}$$

According to the Kirchoff Hypothesis for plates,

$$\epsilon_x{}^s = \epsilon_x + z\kappa_x \tag{18a}$$

$$\epsilon_y{}^s = \epsilon_y + z\kappa_y \tag{18b}$$

$$\epsilon_{xy}{}^s = \epsilon_{xy} + 2z\kappa_{xy} \tag{18c}$$

where ϵ_x, ϵ_y, and ϵ_{xy} are the strains at the reference surface of the plate, and κ_x, κ_y, and κ_{xy} the curvatures. Substituting these expressions in Eq. (17) and integrating with respect to z gives

$$\Pi_S = \frac{1}{2}\int_0^{d_x}\int_0^{d_y}\left\{C\left[\epsilon_x{}^2 + \epsilon_y{}^2 + 2\nu\epsilon_x\epsilon_y + \frac{1}{2}(1-\nu)\epsilon_{xy}{}^2\right] + D\left[\kappa_x{}^2 + \kappa_y{}^2 + 2\nu\kappa_x\kappa_y + 2(1-\nu)\kappa_{xy}{}^2\right]\right\} \times dydx \tag{19}$$

where

$$C = Et/(1 - \nu^2),\ D = Et^3/12(1 - \nu^2)$$

The strain-displacement and curvature-displacement relations for small displacement gradients are

$$\epsilon_x = u_{,x},\quad \kappa_x = w_{,xx} \tag{20a}$$

$$\epsilon_y = v_{,y},\quad \kappa_y = w_{,yy} \tag{20b}$$

$$\epsilon_{xy} = u_{,y} + v_{,x},\quad \kappa_{xy} = w_{,xy} \tag{20c}$$

where u, v, and w are the displacements of the plate in the x, y, and z directions, respectively. Substituting these relations in Eq. (19) gives the strain energy of the skin in terms of displacements as follows:

$$\Pi_S = \frac{1}{2}\int_0^{d_x}\int_0^{d_y}\left\{C\left[u_{,x}{}^2 + v_{,y}{}^2 + 2\nu u_{,x}v_{,y} + (1-\nu)v_{,x}u_{,y} + \frac{1}{2}(1-\nu)v_{,x}{}^2\right] + D\left[w_{,xx}{}^2 + w_{,yy}{}^2 + 2\nu w_{,xx}w_{,yy} + 2(1-\nu)w_{,xy}{}^2\right]\right\} dydx \tag{21}$$

The strain energy of the primary stiffening system can be derived following the procedure used by Meyer.[11] For the stiffening system shown in Fig. 1, the effect of the primary ribs can be averaged, and the stress-strain relations for the primary stiffeners are

$$\sigma_x{}^R = (Et_1/d)\epsilon_x{}^R \tag{22a}$$

$$\sigma_y{}^R = (Et_1/d)\epsilon_y{}^R \tag{22b}$$

$$\sigma_{xy}{}^R = 0 \tag{22c}$$

Using these stress-strain relations in place of the plane stress equations, and proceeding as previously described gives the strain energy of the primary stiffeners in terms of displacements

$$\Pi_R = \frac{Et_1h_1}{2d}\int_0^{d_x}\int_0^{d_y}(u_{,x}{}^2 + v_{,y}{}^2)dydx + \frac{Et_1h_1(t+h_1)}{2d}\int_0^{d_x}\int_0^{d_y}(u_{,x}w_{,xx} + v_{,y}w_{,yy})dydx + \frac{1}{2}I\int_0^{d_x}\int_0^{d_y}(w_{,xx}{}^2 + w_{,yy}{}^2)dydx \tag{23}$$

The work done by the portion of the external load acting on the stiffened panel is

$$\Pi_L = \int_0^{d_x}\int_0^{d_y} N'_2\left(v_{,y} + \frac{1}{2}w_{,y}{}^2 + e_yw_{,yy}\right)dydx \tag{24}$$

where e_y is the eccentricity of the load from the reference surface, and N'_2 is the buckling load of the stiffened panel.

Displacement functions which satisfy the boundary conditions are

$$u = U_{mn}\sin(m\pi y/d_y)\cos(n\pi x/d_x) \tag{25a}$$

$$v = V_{mn}\cos(m\pi y/d_y)\sin(n\pi x/d_x) \tag{25b}$$

$$w = W_{mn}\sin(m\pi y/d_y)\sin(n\pi x/d_x) \tag{25c}$$

where m and n are integers and U_{mn}, V_{mn}, and W_{mn} are unknown displacement amplitudes. Using these in the expressions for Π_S, Π_R, and Π_L, and applying the Theorem of Minimum Potential Energy yields finally

$$N'_2 = kE\left(\frac{\pi}{d_x}\right)^2\left[\frac{t^3}{12(1-\nu^2)} + \frac{C_1}{d}\left(\frac{1}{3}h_1^2 + \frac{1}{2}th_1 + \frac{1}{4}t^2\right)t_1h_1 - \frac{C_2t_1^2h_1^2(t+h_1)^2}{d^2\left(\dfrac{t}{1-\nu^2} + \dfrac{C_1t_1h_1}{d}\right)}\right] \quad (26)$$

where

$$C_1 = \tfrac{1}{2},\ C_2 = \tfrac{1}{16}$$

and n is taken as unity and k is the standard plate buckling coefficient.[6]

Since only a portion of the external load is carried by the stiffened panel, the critical load for panel buckling is

$$N_2 = \frac{\bar{t}}{t + (t_1h_1/d)}N'_2 \quad (27)$$

For plates which may not have secondary stiffeners, it may be noted that the gross buckling load is obtained, using the preceding method, by replacing d_x by a and d_y by b. That is,

$$N^*_2 \equiv N'_2,\ a \rightarrow d_x \quad (28)$$

where N^*_2 denotes the gross buckling load for a plate stiffened by primary ribs only.

3. Skin Buckling

The basic skin panels are square, with uniform thickness t and length $(d - t_1)$. These panels may be conservatively assumed to be simply supported. The buckling load is then given by

$$N'_3 = \left(\frac{\pi t}{d - t_1}\right)^2\frac{C_3Et}{(1-\nu^2)} \quad (29)$$

where $C_3 = \frac{1}{3}$. As in the previous case, the actual failure load becomes

$$N_3 = (\bar{t}/t)N'_3 \quad (30)$$

Failure load for skin buckling in the case of the plates with no secondary stiffening will be given by

$$N^*_3 = \frac{t + (t_1h_1/d)}{t}N'_3 \quad (31)$$

4. Primary Rib Buckling

Both the primary and secondary ribs behave as long panels, free on one long edge and simply supported on the other three edges. For the primary stiffeners[6]

$$N'_4 = \left(\frac{\pi t_1}{h_1}\right)^2\frac{C_4Et_1}{(1-\nu^2)}\left[\left(\frac{h_1}{d-t_1}\right)^2 + 0.456\right] \quad (32)$$

where $C_4 = \frac{1}{12}$. Considering the portion of the load acting on the ribs gives

$$N_4 = (\bar{t}/t_1)\,N'_4 \quad (33)$$

as the buckling load for plates containing both primary and secondary stiffening, and

$$N^*_4 = \frac{t + t_1(h_1/d)}{t_1}N'_4 \quad (34)$$

for plates with primary stiffening only.

5. Secondary Rib Buckling

The analysis for buckling of the secondary stiffeners follows directly from that given above. The failure load is

$$N_5 = (\bar{t}/t_2)N'_5 \quad (35a)$$

where

$$N'_5 = \left(\frac{\pi t_2}{h_2}\right)^2\frac{C_4Et_2}{(1-\nu^2)}\left[\left(\frac{h_2}{d_y}\right)^2 + 0.456\right] \quad (35b)$$

6. Material Yielding

For plates with both primary and secondary stiffening, the external load necessary to cause yielding in uniaxial compression is

$$N_6 = \bar{t}\sigma_y \quad (36)$$

For plates with only primary stiffeners this reduces to

$$N^*_6 = (t + [t_1h_1/d])\sigma_y \quad (37)$$

Optimization Algorithm

Both constrained optimization techniques[1,2,12,13] and penality function techniques[3,14] have been used successfully to solve structural optimization problems. In all of the penality functions employed to date, the multipliers used to add the constraints to the objective function have been somewhat arbitrary in nature, and in some cases[15] the numerical value selected for these multipliers may influence the rate of convergence. For this reason, Fiacco and McCormick have suggested a nontrivial variation[4] for an earlier penality function of theirs[3,15,16] which avoids such multipliers in theory. This new function was used for the work reported in this paper. In practice, however, some arbitrary decisions are still required, as described below.

The general function to be minimized has the form

$$Q(\mathbf{X},\mathbf{X}^k) = \frac{1}{f(\mathbf{X}^k) - f(\mathbf{X})} + \sum_i \frac{1}{g_i(\mathbf{X})} \quad (38)$$

An initial solution, $\mathbf{X}^0$, is chosen such that $g_i(\mathbf{X}^0) > 0$ for all i. Then, minimizing $Q(\mathbf{X},\mathbf{X}^0)$ over $\mathbf{X}$, under suitable conditions,[4] yields the new solution $\mathbf{X}^1$. Replacing $\mathbf{X}^0$ by $\mathbf{X}^1$, the process is repeated, giving $\mathbf{X}^2$, etc. In Ref. 4, it is shown

Table 1 Results of the design of panels containing only primary stiffeners

Load, k/in.	Index, $(N_y)/a$, psi	t, in.	d, in.	t_1, in.	h_1, in.	F_{eq}, ksi	Weight, lb	g_2	g_3	g_4	g_6
1.000	83	0.026	0.999	0.029	0.432	20.06	0.725	0.041	0.005	0.037	1.727
3.000	250	0.046	1.373	0.050	0.564	34.93	1.249	0.031	0.000	0.055	0.763
5.000	417	0.054	1.400	0.062	0.627	46.39	1.567	0.019	0.000	0.023	0.170
7.200	600	0.075	1.841	0.077	0.708	54.14	1.934	0.034	0.000	0.025	0.016
14.400	1200	0.180	4.264	0.117	0.937	62.42	3.355	0.048	0.031	0.077	0.000

Table 2 Results of the design of panels with orthogonal primary stiffeners and one secondary stiffener in both the x and y coordinate directions

Index N_y/a, psi	t, in.	d, in.	t_1, in.	h_1, in.	t_2, in.	h_2, in.	F_{eq}, ksi	wt, lb	g_1	g_2	g_3	g_4	g_5	g_6
83	0.011	0.361	0.016	0.229	0.059	0.597	27.05	0.538	0.009	0.025	0.000	0.016	0.077	0.710
250	0.027	0.680	0.031	0.312	0.095	0.753	45.27	0.964	0.003	0.000	0.020	0.040	0.114	0.091
417	0.051	1.227	0.044	0.386	0.109	0.826	53.93	1.348	−0.001	0.008	0.000	0.054	0.148	0.007
600	0.082	1.991	0.055	0.455	0.117	0.898	57.80	1.812	−0.020	0.000	0.032	0.065	0.116	0.008
1200	0.184	2.705	0.059	0.477	0.131	1.062	63.34	3.307	−0.033	1.635	0.018	0.073	0.016	0.000

that the sequence of points thus generated converges to the solution of the constrained optimization problem. Using this method, unconstrained problem $k + 1$ follows directly from problem k, independent of any parameter adjustment.

The method used to minimize $Q(\mathbf{X}, \mathbf{X}^k)$ for each value of k was the variable metric method as presented by Fletcher and Powel.[17] In this approach, for each unconstrained problem k the gradient of $Q(\mathbf{X}, \mathbf{X}^k)$ is needed for many different values of $\mathbf{X}$. Due to the nature of the expressions for the constraints, it was not convenient to arrive at closed-form equations for this gradient. Instead, the gradient was calculated using a standard forward difference operator.

In finding the optimum value of any unconstrained function by mathematical programing methods, an initial arbitrary choice of the values of the design variables is necessary. Convergence of the optimization algorithm was found to depend somewhat on the initial starting point $\mathbf{X}^0$, and was improved when both terms on the right side of Eq. (38) were equally weighted. After $\mathbf{X}^0$ has been selected for use in Eq. (38), a solution $\mathbf{X}$ is needed, where $f(\mathbf{X}) < f(\mathbf{X}^0)$. In this work new values of $\mathbf{X}$ were computed by taking small steps starting from $\mathbf{X}^0$, in the direction of the negative gradient of $f(\mathbf{X}^0)$. All generated values of $\mathbf{X}$ are restricted to the feasible region and designs very close to constraints were avoided in the computations. This scheme was also used for each subsequent unconstrained problem and was found to work well.

In the minimization of $Q(\mathbf{X},\mathbf{X}^k)$ for unconstrained problem k, the computations must again not produce a design in which any constraint is violated. If this does occur, convergence is not assured and computational difficulties may arise. Such situations were avoided by checking the constraints for each candidate design, and if any constraint became active as an equality during the search procedure, the search was terminated before the constraint was actually violated. A new gradient was then computed for the variable metric method, and a new search was started. If this new direction immediately led to a design in which a constraint was violated, unconstrained problem k was ended and problem $k + 1$ was initiated. The entire process was discontinued when no improvement in the $Q(\mathbf{X},\mathbf{X}^k)$ function could be obtained.

No checks were built into the computer program to test for convergence. Confidence in the final designs was established by running the same problem from several different starting points. All computations were performed on an IBM 360/75 computer, and a typical design took from 5 to 20 sec to complete. The majority of problems all converged in five steps, i.e. after completing five unconstrained minimizations.

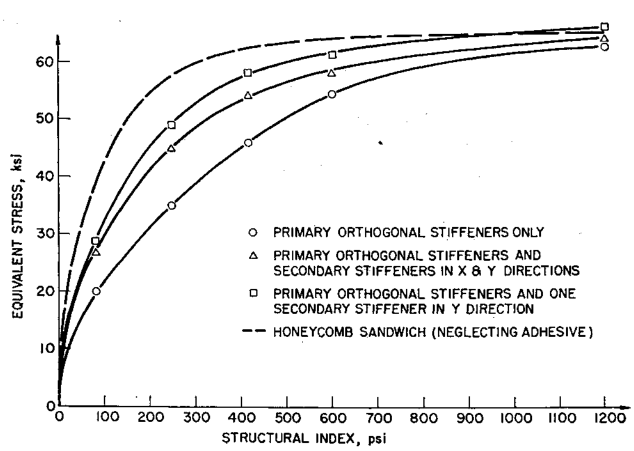

Fig. 2 Structural index vs equivalent stress for example designs.

Examples

In order to investigate the effectiveness of plates with both primary and secondary stiffening, four complete series of example designs were examined. Two sets of plates with different secondary stiffening patterns were compared to plates with primary (basic) stiffeners only, and to minimum-weight designs for sandwich plates subjected to identical ranges of loading intensities, as measured by the "structural index" N_y/a.

Plate dimensions were chosen as $a = b = 12.0$ in., and material properties were selected to correspond to 7075-T6 aluminum alloy with $E = 10.5 \times 10^6$ psi, $\nu = 0.313$, and $\sigma_y = 70$ ksi.

The first series of plates, for which final values of design variables are shown in Table 1, contain only primary stiffeners, and are used as a base for comparing relative efficiencies of other structural concepts. Methods of predicting behavior of these plates have been discussed in previous sections, and it may be noted that only four failure modes are considered. Failure loads correspond to the quantities N^*_2, N^*_3, N^*_4, and N^*_6.

Results are also illustrated in Fig. 2, which shows "equivalent stress" (load divided by effective thickness) as a function of structural index. A high value of equivalent stress at a given structural index indicates an efficient design.

Table 2 shows the resulting designs for plates having pri-

Table 3 Results of parametric study of secondary stiffener configuration at a loading of 1 k/in.

Configuration	F_{eq}, ksi
No secondary ribs	20.1
One secondary rib in y direction	29.0
One secondary rib in x direction	22.3
One secondary rib in x and y directions	27.1
Two secondary ribs in x and y directions	25.2
Three secondary ribs in x and y directions	24.2

Table 4 Results of the design of panels with orthogonal primary stiffeners and one secondary stiffener in the y coordinate direction

Index N_y/a, (psi)	t, in.	d, in.	t_1, in.	h_1, in.	t_2, in.	h_2, in.	F_{2eq}, ksi	wt, lb	g_1	g_2	g_3	g_4	g_5	g_6
83	0.010	0.010	0.330	0.015	0.224	0.074	0.736	29.02	0.501	0.001	0.000	0.018	0.095	0.710
250	0.024	0.604	0.029	0.305	0.116	0.910	49.13	0.888	−0.002	0.022	0.000	0.064	0.137	0.100
417	0.046	1.101	0.042	0.379	0.131	1.012	58.14	1.251	−0.004	0.061	0.000	0.106	0.071	0.008
600	0.073	1.760	0.055	0.450	0.202	1.022	61.34	1.707	0.009	0.000	0.080	0.208	1.496	0.009
1200	0.176	3.402	0.065	0.526	0.201	1.205	66.76	3.137	0.046	0.513	0.018	0.132	0.776	0.000

mary stiffeners plus one centrally located secondary stiffener in both the x and y directions. This configuration was selected for detailed design at all loading intensities after a parametric study of other symmetric arrangements of secondary stiffeners at the lowest loading intensity proved it to be the most efficient. The results of this study are shown in Table 3. From Fig. 2 it is seen that the designs for this configuration are all significantly superior to the designs which do not include secondary ribs.

Table 4 shows designs for plates having primary stiffeners plus one central secondary stiffener in the y direction only. As shown in Fig. 2, this configuration was found to be slightly more efficient than the preceding one. Again, this result was obtained by a parametric study of the effects of varying the dimensions d_x and d_y. Until these quantities, as well as the spacing of primary ribs in each direction, are introduced into the synthesis procedure as variables, there will be some question as to what constitutes the actual optimum configuration. Nevertheless, the potential improvement in efficiency achievable through use of secondary ribs is well demonstrated by these examples.

For all designs, all behavioral constraints, except yielding. are very nearly active at low loading intensities, as should be anticipated for single load conditions. Final values of g_i for $i = 1,2, \ldots, 6$ are listed in Tables 1, 2, and 4.

Finally, for purposes of comparison, equivalent stresses for optimum designs of sandwich plates of identical material are illustrated in Fig. 2. The curve of equivalent stress vs structural index for honeycomb core construction is based on the work of Kaechele.[18] The sandwich, as expected, is the most efficient type of construction at low load intensities, but the "pyramided" stiffening of the waffle plate has reduced the weight difference between sandwich and "basic" waffle by approximately 50% in the region of interest.

Conclusion

It has been shown that the inclusion of multiple levels of rib sizes in the design of waffle plates may lead to significant improvements in structural efficiency. Also, mathematical programing techniques, and particularly the Fiacco-McCormick penalty function technique, proved capable of efficiently coping with synthesis problems of this type.

It should be re-emphasized that the designs examined in the previous section are not represented as absolute optimum configurations in either a geometric or computational sense, but are, rather, intended to provide an indication of the potential value of the structural concept under investigation. Future studies should be directed toward improving the capability of the analysis and design techniques in order to include additional levels of design variables, including basic geometric orientation of stiffeners.

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